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REVIEWS

Free and Moving Boundary Problems. By J. Crank. Clarendon, Oxford, 1984. 425 pp. £45.00.

Modern applied mathematics is very concerned with the solution of partial differential equations by both analytical and numerical methods, especially when the equations model an interesting physical situation. Since explicit analytical solutions are rare the emphasis is necessarily on the formulation of well-posed problems, that is problems for which a unique solution exists which is continuously dependent on the data, so that there is some hope of finding convergent numerical procedures for their solution. For linear partial differential equations in two independent variables there are well established results, especially for second order equations, about the appropriate boundary or initial data needed to make the problem well-posed; for nonlinear equations very few results are available.

An interesting class of problems which are only weakly nonlinear are those for which the equation is linear but there is an unknown curve either bounding the domain or defining a line of discontinuity in the variables. These problems may be called moving boundary problems if the two independent variables are space and time, or free boundaries if they both are spatial variables, although in more than two independent variables when the boundaries become surfaces this terminology loses its clarity. Professor Crank's welcome book discusses such problems and makes great distinction between free and moving boundaries. However his free boundaries bound domains in which the governing equation is elliptic and his moving boundaries are associated with parabolic equations, and it is surely this feature which is the fundamental reason why different methods of approach are needed in the two cases. Such is the diversity of these methods that there is no room for any discussion of moving (or free) boundaries for a hyperbolic equation such as occurs in gas dynamics, where, however, the equation is nonlinear, and problems with more than two independent variables are discussed only briefly and in the context of the three-dimensional Laplace equation and the two-dimensional diffusion equation. Thus the applications are focused on the classical problem of seepage through a dam and the Stefan phase-change problem, together with their generalisations and degeneracies.

The plan of the book, which is beautifully produced and written in a very clear style, is first to formulate the two classical problems in Chapters 1 and 2 respectively and to discuss the formulation of a wide range of generalizations of these problems. Then follow four chapters on analytical and numerical methods used in free boundary problems, and finally two chapters on analytical and numerical methods used for moving boundary problems. This breakdown makes clear Professor Crank's main purpose which is to give a clear and honest account of practical methods for obtaining useful solutions, and the reader can be confident that the author has tried and tested almost all of them. Thus for the theoretical engineer or applied numerical analyst this book provides an entrée to free and moving boundary problems which previously did not exist except piecemeal in widely scattered journals. There is an excellent range of references given for the various methods, although it is a little surprising that Chapter 5 of Yih's book, *Dynamics of non-homogeneous fluids*, is omitted.

For the applied mathematician, however, there are some disappointments. The generalizations of the Stefan problem, such as density changes and convection, ablation, phase change of mixtures, diffusion with sources, concentrated thermal capacities, more space dimensions and more phases, are dismissed in twenty pages of Chapter 1 and most of them never mentioned again. A difficult idea such as the ill-posedness of some Stefan problems with volumetric sources and the consequent introduction of mushy regions is dismissed in a few sentences and the importance of the lack of stability of the moving boundary for negative latent heat and in the alloy problem is ignored. The generalization of the dam-seepage problem includes, in addition to different geometries and the stratified dam, the degenerate problems associated with Hele-Shaw flows. Again the stability of the free boundary is not discussed and the non-trivial ideas of variational inequalities, no doubt new to many readers, only receive two pages of explanation. These are clearly all compromises which the author has thought about but they do make the first two chapters hard to read and could easily deter the inexperienced reader from attempting the more straightforward chapters on analytical and numerical methods. These later chapters contain much material which would be valuable in a graduate course but it is not clear to me whether Professor Crank had such readers in mind; and if he did a more leisurely introduction with schematic (x,t) diagrams of boundary-value problems might have been more appropriate. If, however, his audience was intended to be more experienced researchers then more detail, possibly at the end, about the fascinating generalizations of the two classical problems would surely have been a valuable and satisfying addition.

There is however no alternative book available which draws together such a range of practical methods for moving- and free-boundary problems and it would grace the shelf of any applied mathematician or theoretical engineer.

A. B. TAYLER

Numerical Methods for Nonlinear Variational Problems. BY R. GLOWINSKI. Springer, 1984. 493 pp. DM 158.00.

As suggested by the title, this book deals with several methods for the numerical solution of nonlinear problems of mechanics. It is not only concerned with fluid mechanics but the numerical solutions of a number of problems in this field are treated in depth: visco-plastic non-Newtonian fluids (Bingham flows), subsonic and transonic potential flow, Stokes flow and incompressible viscous flows (Navier–Stokes equation) at moderate Reynolds number.

The numerical methods are presented in the framework of Sobolev spaces, the variational form for the partial differential equations, and their discretization by finite-element methods. In general, the nonlinear problems considered in this book are first studied from a mathematical point of view (existence and uniqueness of solutions), then converted into a sequence of linear or quasi-linear problems by a number of techniques: penalization; relaxation; augmented Lagrangian; least squares; conjugate-gradient methods... The presentations are given with full details and convergence proofs whenever available. The first four chapters are devoted to problems which can be converted into variational inequalities (Bingham flows and subsonic potential flows are in that category). The final chapters deal with general methods (relaxation, decomposition-coordination and least-squares methods) and the chapter on least squares deals at length with the numerical solution of the transonic equation and of the Navier–Stokes equation. Finally, two long appendices are devoted to the approximation of the Navier–Stokes equation by the finite-element method.

The book reads well and the author is certainly a well-known expert in the field of numerical analysis; in fact he has personally contributed to most of the topics presented. It is a useful book for anyone who seeks a numerical solution of a nonlinear problem because most of the methods presented are actually used in industry. However, the reader must have an elementary background in variational methods and in numerical analysis because it is more a reference book than a beginner's book.

O. PIRONNEAU

Mathematical Structure of the Singularities at the Transitions between Steady States in Hydrodynamical Systems. By H. N. SHIRER and R. WELLS. Lecture Notes in Physics vol. 185, Springer, 1983. 276 pp. DM 29.00.

Transitions in hydrodynamical systems have been the focus of much recent research, motivated to a large extent by the perennial problem of turbulence. While little has been learned about turbulence much important and interesting work has been done in the process. But the importance of this subject to hydrodynamics goes beyond the mere accumulation of new results, for in one way it is likely to have a long-lasting effect on fluid dynamics. This is because it has led to an interest in fluid dynamics on the part of (some) mathematicians, and an interest in learning new developments in mathematics on the part of (some) fluid dynamicists. This is a trend that is to be welcomed, for fluid dynamicists have a great deal to learn from mathematicians, especially those that are prepared to break the communication barrier that has steadily been worsening.

Over the last twenty years or so a number of new techniques for studying nonlinear phenomena have become available that remain unfamiliar to fluid dynamicists, in spite of being more powerful than perturbation theory, the traditional tool of the theorist. This is primarily because perturbation theory is designed to discover solutions of specific form, and will not locate solutions of an unknown one. But more important is the evolution of a new point of view that emphasizes highly degenerate bifurcations as the key to the understanding of the complexity of behaviour in a system. At those parameter values multiple instabilities occur simultaneously, and typically the solutions at nearby parameter values capture in a qualitative way much of the nonlinear behaviour of the system even at substantially different parameter values. In a branch of mathematics called *singularity theory* dealing with steady-state bifurcations, this approach is formalized in the search for an *organizing centre*, i.e. parameter values yielding the most degenerate bifurcation (singularity) with the property that all other simpler bifurcations in the system are found in the *unfolding* (i.e. breaking apart) of this degeneracy. The theory describes the most general way of unfolding a degeneracy, and thus classifies the possible behaviour of the system. In this the theory is capable of describing the qualitative effects of imperfections in the system that would be hard to establish without detailed and laborious calculations – once the imperfection was identified! For example, the theory shows that the effects of a variety of imperfections in a convection experiment (non-Boussinesq fluid, sideways heat flux, non-parallel walls, tilted apparatus, radiation from the surface, etc.) can all be lumped into two parameters, in addition to the Rayleigh number, and thus no new phenomena will be found by studying these effects separately.

It is regrettable that the volume under review does not succeed in making this important subject accessible to fluid dynamicists. While the formal theory, presented in Chapters 2 and 6, is simplified for the consumption of fluid dynamicists by the omission of proofs, it is not well motivated and remains couched in a parlance that fluid dynamicists will find unappetizing. The theory employs so-called contact transformations to simplify as much as possible the equations for the steady states without destroying them. The technique has the advantage that typically only a few unfolding parameters are required to capture all the nearby steady states, but suffers from the disadvantage that the transformation does not preserve stability assignments. In fact in most cases stability can be determined by *ad hoc* methods, although the space of transformations can be restricted in order to preserve stability. This happens automatically when the equations are transformed as differential equations, but typically requires many more unfolding parameters. The theory that is described is a generalization of Thom's catastrophe theory, and treats all the parameters on an equal footing. Of greater interest to fluid dynamicists is a version of the theory with a distinguished parameter, which is capable of describing the most general effect of imperfection on the succession of transitions as a parameter such as the Rayleigh number is increased. This is of direct relevance to the vast body of experimental data which, like the theory, is scarcely mentioned. In Chapters 3–5 the authors use the much abused Lorenz equations (and two other examples of the same type) to illustrate their results, without making clear that these equations are valid only near the onset of two-dimensional convection, and thereby endow bifurcations at large values of the Rayleigh number with a spurious significance. This is unfortunate because the value of the theory lies in describing the generic break-up of the pitchfork bifurcation describing the onset of convection. The authors do not describe how to apply their techniques to partial differential equations.

An excellent book on this subject, *Singularity Theory and Applications*, by M. Golubitsky and D. Schaeffer (Springer-Verlag 1984) should be available by the time this review appears.

E. KNOBLOCH